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ELASTIC–PLASTIC ANTI-PLANE PROBLEMS FOR BONDED DISSIMILAR MEDIA CONTAINING CRACKS AND CAVITIES*

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Abstract—The problem of two bonded semi-infinite media with different properties containing cracks or cavities on the interface and subjected to longitudinal shear loads in various ways is considered. The elastic solution in which only the z-component of the external loads is taken into account, complements the plane strain problem where the external loads lie in xy-plane. Using the approach of the dislocation theory to plastic deformations, a simple technique based on the elastic theory is developed to estimate the size of the plastic zone in the neighborhood of cracks and cavities. Some examples are worked out and a list of solutions is given in the Appendix.

NOTATION

$\pm a, a_k, b_k$	end points of bonds or cracks
$g_k(t)$	tractions on the crack surface
$h(t)$	relative displacement on $y = 0$
$h(a)$	crack tip displacement
$k = \lim_{t\to a} \sqrt{(t-a)}\tau_{yz}(t)$	stress intensity factor
$L = \Sigma L_j$	bonding segments
L'	complement of L
$q_0 = \tau_{yz}^{\infty}$	external load
p	plastic zone size
$x, y, z; r, s, \sigma$	cartesian coordinates
$\pm\alpha, \alpha_j, \beta_j$	end points of plastic zones
μ_1, μ_2	shear moduli
$\eta = r+is, \zeta = x+iy$	complex variables
$\omega(\zeta)$	mapping function
Ω_1, Ω_2, G	sectionally holomorphic functions
τ_0	yield stress on the interface
$\tau_{kyz}, \tau_{kxz}, \tau_{k\sigma s}, \tau_{k\sigma r}$	shear stresses

1. INTRODUCTION

RECENT interest and developments in fracture mechanics have attracted considerable attention to the calculation of stress singularities and plastic deformations in materials containing cracks, sharp notches and other stress raisers (for survey and references, see: [1–3]). There have also been some studies on the singular character of the stresses [4, 5], as well as the stress distributions [6, 7] in bonded dissimilar media with cracks. The elastic anti-plane problem, or the problem of longitundinal shear of homogeneous cylindrical

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bodies has been considered, among others, in [8 through 11]. Aside from its direct application to the torsion of cylindrical bars, the solution of this problem complements that of plane strain where all the external loads, which are independent of z, lie in the xy-plane. In [8] and [9] the problem is solved by mapping the infinite plane containing the imperfection outside the unit circle and using a complex potential whereas in [10] and [11] the approach of dislocations is used.

If the stress raiser is a sharp notch with an angle $\alpha < \pi$, the stresses will have a singularity at the vertex of the notch which in anti-plane problems is of the order of $r^{-[(\pi-\alpha)/(2\pi-\alpha)]}$, r being the distance from the vertex. Thus, in the presence of sharp notches as well as the stress raisers with finite radius of curvature under sufficiently high loads, some plastic deformations will take place in the neighborhood of the points of singularity or stress concentration. In [12–15] the plasticity problem is treated by using the field approach with maximum shear as the yield criterion [12, 13, 14], or a nonlinear stress–strain relation [15]. The solution given in [12] is significant in that it contains numerical results based on an exact solution. In [16–19] the anti-plane plasticity problem for an infinite homogeneous medium with cracks is treated by using the approach of dislocations. It is assumed that the plastic deformations in the neighborhood of crack tips may be represented by an array of screw dislocations coplanar with the crack.

In this paper we first consider the general anti-plane elastic problem for two bonded dissimilar media containing cracks or cavities on the interface. This solution will complement the plane strain problem for bonded materials given in [6] and [7]. After pointing out the equivalency of the formulations obtained through the use of dislocations and that of this paper, we next consider the general plasticity problem for the bonded dissimilar media with cracks and/or cavities subjected to longitudinal shear loads in various ways. A simple expression for the plastic zone size in terms of stress intensity factor valid for small ratios of applied-to-yield stress is derived. The notch-root-displacement fracture criterion used in [16–19] is considered for the bonded materials, its equivalency to Irwin–Orowan theory is shown and a simple form of the criterion in terms of stress intensity factor is derived. It is indicated that for $\mu_1 = \mu_2$, μ_i being the shear moduli of the bonded media, the results of this paper reduce to those found for homogeneous materials whenever available [e.g. 9, 10, 11, 16].

2. THE ELASTIC THEORY

Consider an infinitely long cylindrical elastic body with its generators parallel to z direction. Let the specified external loads and displacements be parallel to the z axis and independent of the z coordinate. If μ is the shear modulus, the displacements and stresses may be written as

$$u = 0, \qquad v = 0, \qquad w = w(x, y),$$
$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0, \qquad \tau_{yz} = \mu\frac{\partial w}{\partial y}, \qquad \tau_{xz} = \mu\frac{\partial w}{\partial x}. \tag{1}$$

The equilibrium conditions give $\nabla^2 w = 0$ and in terms of an analytic function $f(\zeta)$, $\zeta = x+iy$, we may write

$$w = \operatorname{Re} f(\zeta), \qquad \tau = \tau_{xz}+i\tau_{yz} = \mu\overline{f'(\zeta)}. \tag{2}$$

Now, let two semi-infinite elastic media with shear moduli μ_1 and μ_2 occupy the lower and upper-half planes S^- and S^+, respectively and be bonded along the strips $L_1, \ldots, L_n$ on the x axis with $L = \Sigma^n L_k$. Let $L' = \Sigma L'_k$ be the complement on L with L'_k representing the cracks on the interface. In addition to the loads at infinity consider the following general boundary conditions:

$$\begin{aligned} w_1^- - w_2^+ &= h(t), \qquad t \in L \\ \tau_{1yz}^-(t) &= g_1(t), \qquad t \in L' \\ \tau_{2yz}^+(t) &= g_2(t), \qquad t \in L' \end{aligned} \tag{3}$$

where h, g_1, g_2 are known functions, subscripts 1 and 2 refer to lower and upper-half planes and t is the coordinate along the real axis. Let the functions $f_1(\zeta)$ and $f_2(\zeta)$ be analytic in S^- and S^+, respectively. Defining the functions Ω_1 and Ω_2 we extend the definition of f'_1 and f'_2 into S^+ and S^- in such a way that they are holomorphic on the unloaded parts of the real axis:

$$\Omega_1(\zeta) = \begin{cases} f'_1(\zeta) & \zeta \in S^- \\ \bar{f}'_1(\zeta) & \zeta \in S^+ \end{cases}, \qquad \Omega_2(\zeta) = \begin{cases} f'_2(\zeta) & \zeta \in S^+ \\ \bar{f}'_2(\zeta) & \zeta \in S^- \end{cases}. \tag{4}$$

Noting that $\tau_{1yz}^-(t) = \tau_{2yz}^+(t)$ for $t \in L$ and using second and third equations of (3) we may write

$$\begin{aligned} [\mu_1\Omega_1^+(t) + \mu_2\Omega_2^+(t)] - [\mu_1\Omega_1^-(t) + \mu_2\Omega_2^-(t)] &= 2i(g_1 - g_2) \quad t \in L' \\ &= 0 \qquad\qquad t \in L \end{aligned} \tag{5}$$

which gives

$$\mu_1\Omega_1(\zeta) + \mu_2\Omega_2(\zeta) = \frac{1}{2\pi i}\int_{L'} \frac{2i(g_1 - g_2)}{t - \zeta}\,dt + P(\zeta) = H(\zeta) \tag{6}$$

where $P(\zeta)$ is an arbitrary analytic function consistent with the behavior of Ω_1 and Ω_2 at infinity and is zero if the stress state at infinity vanishes. Substituting from (6), first and second equations of (3) become

$$\begin{aligned} \Omega_1^+(t) + \Omega_1^-(t) &= \frac{2\mu_2}{\mu_1 + \mu_2} h'(t) + \frac{H^+(t) + H^-(t)}{\mu_1 + \mu_2} = p(t), \quad t \in L \\ \Omega_1^+(t) - \Omega_1^-(t) &= \frac{2i}{\mu_1} g_1(t), \qquad t \in L'. \end{aligned} \tag{7}$$

First we give the solution of the Hilbert problem, (7), for the case of finite L. At infinity since the stress state vanishes by (2) and (4) Ω_1 and Ω_2, and as a result, the arbitrary polynomial $P(\zeta)$ will be zero. If the bonds L_k have the finite ends a_k, b_k, $(k = 1, \ldots, n)$, and if we define

$$R(\zeta) = \prod_1^n (\zeta - a_k)^{-\frac{1}{2}}(\zeta - b_k)^{-\frac{1}{2}}. \tag{8}$$

The solution of (7) may be written as [20]

$$\Omega_1(\zeta) = \frac{R(\zeta)}{2\pi i}\int_L \frac{p(t)\,dt}{R(t)(t - \zeta)} + \frac{R(\zeta)}{2\pi i}\int_{L'} \frac{2i}{\mu_1}\frac{g_1(t)\,dt}{R(t)(t - \zeta)} + N(\zeta)R(\zeta) \tag{9}$$

where $N(\zeta)$ is an arbitrary polynomial consistent with the behavior of $\Omega_1(\zeta)$ at infinity. In (9) the particular branch of $R(\zeta)$ for which $\lim_{\zeta\to\infty} \zeta^n R(\zeta) = 1$ is considered and the branch cut is taken along L, hence $R^+(t) = R^-(t)$, $t \in L'$, and $R^+(t) = -R^-(t)$, $t \in L$, with $R(t) = R^+(t)$.

Since Ω_1 vanishes at infinity $N(\zeta)$ may be written as

$$N(\zeta) = A_{n-1}\zeta^{n-1} + \ldots + A_0. \tag{10}$$

From (9) and (10) it follows that as $\zeta \to \infty$ Ω_1 behaves as A_{n-1}/ζ. Observing that the resultant force, T, acting on an arc AB in S^- may be expressed as

$$T = \int_A^B (\tau_{1xz}\,\mathrm{d}y - \tau_{1yz}\,\mathrm{d}x) = \mathrm{Im}\int_A^B \mu_1 f_1'(\zeta)\,\mathrm{d}\zeta$$

the constant A_{n-1} becomes

$$A_{n-1} = -\frac{T}{\pi\mu_1} \tag{11}$$

where T is now the resultant force acting on S^- at infinity. The remaining $n-1$ constants in (10) are determined from the single-valuedness of displacements. The first equation of (7) is the statement of the continuity of the derivatives of w_1 and w_2 along L; i.e. it implies that $w_1^- - w_2^+ = h(t) + d_k$, $t \in L_k$, $k = 1, \ldots, n$, where the arbitrary constants d_k must all vanish. Arbitrarily fixing the first to be zero, the conditions $d_2 = \ldots = d_n = 0$ become

$$\int_{b_{k-1}}^{a_k} [\Omega_1^+(t) + \Omega_1^-(t)]\,\mathrm{d}t = \int_{b_{k-1}}^{a_k} p(t), \qquad (k = 2, \ldots, n) \tag{12}$$

which gives a system of $n-1$ linear equations in $A_{n-2}, \ldots, A_0$.

After $\Omega_1(\zeta)$ is determined, equations (2) with (4) and (6) gives the solution. In particular the contact stresses on the bonds are obtained from

$$\tau_{1yz}^-(t) = \frac{i\mu_1}{2}[\Omega_1^-(t) - \Omega_1^+(t)], \qquad t \in L. \tag{13}$$

If L is infinite we let a_k, b_k stand for the finite ends of the crack L_k', $(k = 1, \ldots, n)$, and define a function $R(\zeta)$ as in (8) with the same properties. Now defining a sectionally holomorphic function $G(\zeta)$, by taking the branch cut along L' instead of L as

$$G(\zeta) = \begin{cases} \Omega_1(\zeta) & \zeta \in S^- \\ -\Omega_1(\zeta) & \zeta \in S^+ \end{cases} \tag{14}$$

the solution may be written as

$$G(\zeta) = -\frac{R(\zeta)}{2\pi i}\int_L \frac{p(t)\,\mathrm{d}t}{R(t)(t-\zeta)} - \frac{2i}{\mu_1}\frac{R(\zeta)}{2\pi i}\int_{L'} \frac{g_1(t)\,\mathrm{d}t}{R(t)(t-\zeta)} + M(\zeta)R(\zeta) \tag{15}$$

where

$$R(t) = R^+(t), \qquad R^-(t) = R(t) \quad t \in L, \qquad R^-(t) = -R^+(t) \quad t \in L'$$

and since the stress state at infinity is bounded the arbitrary polynomial $M(\zeta)$ is, at the most, of degree n:

$$M(\zeta) = A_n\zeta^n + \ldots + A_0 \tag{16}$$

The constant A_n is determined from the stress state at infinity. For example if $\tau^{\infty}_{1xz} = \tau^{\infty}_{2xz} = 0$, $\tau^{\infty}_{1yz} = \tau^{\infty}_{2yz} = q$ = constant we have $A_n = -iq/\mu_1$. The remaining constants in (16) are again determined from the condition of single-valuedness of displacements which, by using a similar argument as before, may be written as

$$\int_{a_k}^{b_k} [G^+(t) - G^-(t)]\,dt = -\int_{a_k}^{b_k} p(t)\,dt, \qquad k = 1,\dots,n. \tag{17}$$

The solution given by (9) indicates that in the neighborhood of a crack tip, b_j, we may write

$$f'_1(\zeta) = F(\zeta)(\zeta - b_j)^{-\frac{1}{2}}$$

$F(\zeta)$ being a holomorphic function. Defining $\zeta - b_j = \rho e^{i\theta}$, $-\pi < \theta < 0$, from (2) it follows that for small values of ρ the stresses may be expressed as*

$$\tau_{1xz} = \mu_1 F(b_j)\rho^{-\frac{1}{2}}\cos\frac{\theta}{2}, \qquad \tau_{1yz} = \mu_1 F(b_j)\rho^{-\frac{1}{2}}\sin\frac{\theta}{2}. \tag{18}$$

The constant $F(b_j)$ depends on μ_2, (i.e. the stresses in S^- are affected by the difference between the shear moduli of the adjoining media), only if the external loads are not symmetric with respect to xz-plane (see the examples and the conclusions). The constant $k = \mu_1 F(b_j)$ is known as the stress intensity factor at b_j.

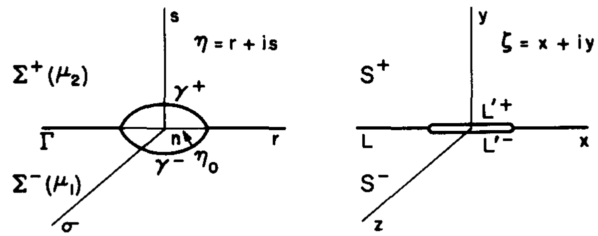

FIG. 1. Bonded plane with cavity mapped on a plane with slit.

The procedure outlined above can also be used to solve the problem of two bonded semi-infinite media containing cavities rather than cracks on the interface provided the cavities are symmetric with respect to $r\sigma$-plane (Fig. 1) and the proper mapping functions can be found. Referring to Fig. 1, the problem may be formulated as

$$\begin{aligned} &\nabla^2 w_1(r,s) = 0 \quad \eta \in \Sigma^-, \qquad \nabla^2 w_2(r,s) = 0 \quad \eta \in \Sigma^+, \\ &w_1^-(r) - w_2^+(r) = F(r) \quad r \in \Gamma \\ &\tau^-_{1n\sigma}(\eta_0) = p_1(r,s) \quad \eta_0 \in \gamma^- \\ &\tau^+_{2n\sigma}(\eta_0) = p_2(r,s) \quad \eta_0 \in \gamma^+ \end{aligned} \tag{19}$$

with the proper conditions at infinity. Assume that an analytic function $\eta = \omega(\zeta)$ is found which conformally maps the Σ plane into S plane in such a way that Σ^+, Σ^-, γ^+, γ^-, Γ are mapped onto S^+, S^-, L'^+, L'^-, L and $\omega(\zeta) \to \zeta$ as $\zeta \to \infty$. The harmonic functions

* For a similar result in the general mixed boundary-value problems see [21] theorem 2.

w_1, w_2 will remain harmonic in $\zeta = x+iy$ plane. Noting that on a surface with outward normal n in rs-plane the shear stress acting in σ direction is given by

$$\tau_{n\sigma} = \mu \frac{\partial w}{\partial n}$$

and using the definition of directional derivatives (19) may be written as

$$\begin{aligned} &\nabla^2 w_1(x,y) = 0 \quad \zeta \in S^-, \qquad \nabla^2 w_2(x,y) = 0 \quad \zeta \in S^+, \\ &w_1^-(t) - w_2^+(t) \doteq h(t) \quad t \in L \\ &\left(\mu_1 \frac{\partial w_1}{\partial y}\right)^- = |\omega'(t)| g_1(t) \quad t \in L' \\ &\left(\mu_2 \frac{\partial w_2}{\partial y}\right)^+ = |\omega'(t)| g_2(t) \quad t \in L' \end{aligned} \tag{20}$$

where functions $g_1(t)$, $g_2(t)$ and $h(t)$ are obtained from the known functions p_1, p_2 and F. We call (20) the "equivalent longitudinal shear" problem the solution of which is obtained from (9) or (15) by simply replacing g_1 and g_2 by $|\omega'|g_1$ and $|\omega'|g_2$. Note that, since $\omega(\zeta) \to \zeta$ as $\zeta \to \infty$, $(\tau_{kr\sigma} + i\tau_{ks\sigma}) \to (\tau_{kxz} + i\tau_{kyz})$, $k = 1, 2$, as $\zeta \to \infty$ and the contact stresses may be written as

$$\tau_{1s\sigma}^-(r) = \mu_1 \left(\frac{\partial w_1}{\partial s}\right)^- = \frac{1}{|\omega'(t)|} \tau_{1yz}^-(t), \qquad r \in \Gamma, \qquad t \in L. \tag{21}$$

3. RELATION TO DISLOCATIONS AND PLASTIC DEFORMATIONS

For the semi-infinite medium $y < 0$ subjected to longitudinal shear under a line load T (per unit thickness) acting at $y = 0$, $x = t_0$ the solution may be written as

$$f_1'(\zeta) = \frac{T}{\pi\mu_1(t_0 - \zeta)}$$

from which we obtain

$$w_x^-(t) = \frac{T}{\pi\mu_1(t_0 - t)}$$

where $w_x^-(t)$ is the value of $\partial w/\partial x$ on the real axis as y goes to zero from below. Now, instead of acting at a point if the external loads are distributed on the real axis with a density $g_1(t_0)$ which is square-integrable in $(-\infty, \infty)$, we may write

$$w_x^-(t) = \int_{-\infty}^{\infty} \frac{g_1(t_0)\,dt_0}{\pi\mu_1(t_0 - t)}. \tag{22}$$

Taking the Hilbert transform of (22) we obtain

$$\frac{g_1(t)}{\mu_1} = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{w_x^-(t_0)\,dt_0}{t_0 - t} \tag{23}$$

If we now consider a homogeneous infinite medium containing a series of cuts $L'_1, \ldots, L'_n$ along the real axis on the surfaces of which the shear traction $g_1(t)$ is applied, because of symmetry on the remainder L of the real axis we have $w_x^- = w_x^+ = 0$ and (23) becomes

$$\frac{g_1(t)}{\mu_1} = -\frac{1}{\pi}\int_{L'} \frac{w_x^-(t_0)\,\mathrm{d}t_0}{t_0 - t} \tag{24}$$

This is the well-known equation for the medium containing an array of screw dislocations along the real axis assumed to be smeared over L' and having a density $w_x^+ - w_x^- = -2w_x^-$ [10, 11]. If g_1 is given, it can easily be shown that the solution of the integral equation (24), and hence, the problem may be written as

$$\begin{gathered} w_x^-(t) = -\tfrac{1}{2}(G^+ - G^-) = G^-(t) \\ \frac{1}{\mu_1}(\tau_{xz} - i\tau_{yz}) = G(\zeta) = \frac{1}{2\pi i}\int_{L'} \frac{-2w_x^-\,\mathrm{d}t}{t - \zeta} \end{gathered} \tag{25}$$

where $G(\zeta)$ is obtained from (15) by substituting $g_2 = g_1$ and $\mu_2 = \mu_1$. It is seen that this is a special case of the more general problem considered in the previous section.

Noting that the crack tips, a_k and b_k, are points of stress singularity in the neighborhood of these points plastic deformations are expected to take place. If we adopt the model for these deformations as being the result of an array of screw dislocations distributed ahead of the crack tip on the slip plane which is coplanar with the crack and assume that slip takes place under a constant yield stress in shear, the size of the plastic zone can easily be calculated [16–19]. This is done by writing the condition of equilibrium for the dislocations, i.e. requiring that the resultant force on any dislocation in the distribution is zero [22]. In the terminology of continuum mechanics, this procedure is completely equivalent to the following: If a_j, b_j ($b_j > a_j$) are the crack tips and α_j, β_j refer to the ends of plastic zones (on the x axis) α_j, β_j are determined from the requirement that in the medium cut along $\alpha_j < x < \beta_j$, ($j = 1, \ldots, n$, the stress intensity factors $k(\alpha_j)$, $k(\beta_j)$ calculated from the external loads and from the constant tractions $\tau_{yz}^+ = \tau_{yz}^- = \tau_0$ (τ_0 being the yield stress in shear) applied along $\alpha_k < x < a_k$, $b_k < x < \beta_k$ in the direction opposing the external loads cancel each other. The defect of this idealized model that the displacement is not continuous in the plastic zone may be overcome by assuming that ahead of the crack the slip takes place in a thin strip, $-\delta < y < \delta$. If the strip is thin enough it would not have any significant effect on the calculation of the plastic zone size based on the distributed screw dislocations and, by specifying its thickness, the magnitude of plastic strains may be calculated.

Thus establishing that in longitudinal shear problems the dislocation approach to plastic deformations is equivalent to a simple continuum approach requiring only elastic solutions, we may now free ourselves from the restrictions imposed by the former approach as to the symmetry in loading and geometry, and the homogeneity of the medium. Particularly it now becomes possible to calculate the plastic deformations (in the weaker of the base media or the bond material) in the bonded dissimilar materials containing cracks or cavities on the interface. Furthermore, if the bond material (or a thin layer of composite material on the interface) has the lowest yield stress in shear, knowing the thickness and the total relative displacement, $w^+ - w^-$, the plastic strains and the values of external loads corresponding to rupture strain can be evaluated.

4. EXAMPLES

As a simple example consider the case of the bonded media with a central crack, $L' = (-a, a)$. Let the stress state at infinity and the relative displacement $h(t)$ on L be zero and the shear stress on the crack surface be $g(t) = q_0$ = constant. From (15), (16) and (17) we obtain

$$G(\zeta) = -\frac{iq_0}{\mu_1}[1-\zeta R(\zeta)], \qquad R(\zeta) = (\zeta^2-a^2)^{-\frac{1}{2}}$$

and (13) and (14) give the contact stress as

$$\tau_{1yz}^-(t) = q_0\left[1-\frac{t}{\sqrt{(t^2-a^2)}}\right], \qquad (t > a). \tag{26}$$

By superimposing a uniform shear $\tau_{1yz} = q_0$ on the negative of (26), the contact stress for the bonded plane under uniform shear at infinity, $\tau_{kyz}^{\infty} = q_0$, and free of tractions on the crack surface may be obtained as*

$$\tau_{1yz}^-(t) = \frac{q_0 t}{\sqrt{(t^2-a^2)}}, \qquad (t > a). \tag{27}$$

To obtain the plastic zone size, $\alpha - a$ (Fig. 2), we assume that the crack extends to $\pm\alpha$ and write the total stress intensity factor obtained for the loads q_0 and yield value of the shear τ_0 (corresponding to the weaker of the joining media or the bonding agent) acting on the part of the crack surfaces, $-\alpha < t < -a$ and $a < t < \alpha$, equal to zero.

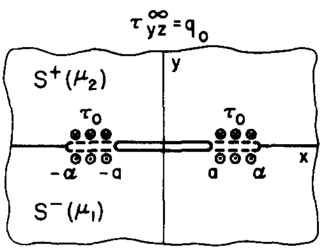

FIG. 2. Bonded plane with central crack.

For the loading by τ_0 (15) gives

$$G(\zeta) = \frac{i\tau_0}{\pi\mu_1}\left[\frac{2}{\sqrt{(\zeta^2-\alpha^2)}}\left(\frac{\pi}{2}-\arcsin\frac{a}{\alpha}\right)-\pi-i\log\frac{(c-\gamma)(1+c\gamma)}{(c+\gamma)(1-c\gamma)}\right]$$

$$c = [(\alpha+a)/(\alpha-a)]^{\frac{1}{2}}, \qquad \gamma = [(\alpha+\zeta)/(\alpha-\zeta)]^{\frac{1}{2}}$$

$$\tau_{1yz}^-(t) = -\frac{\tau_0}{\pi}\left\{\frac{2t}{(t^2-\alpha^2)^{\frac{1}{2}}}\left(\frac{\pi}{2}-\arcsin\frac{a}{\alpha}\right)-\pi\right.$$
$$\left.-2\arctan\left[\frac{(t+\alpha)(\alpha+a)}{(t-\alpha)(\alpha-a)}\right]^{\frac{1}{2}}+2\arctan\left[\frac{(t+\alpha)(\alpha-a)}{(t-\alpha)(\alpha+a)}\right]^{\frac{1}{2}}\right\}. \tag{28}$$

* In this as well as other examples given in this paper note that the sign of $R(t)$ changes as t goes over each branch cut.

Replacing a by α in (27) and adding to (28) we obtain the contact stress corresponding to elastic–plastic deformations. Defining the stress intensity factor as

$$k = \lim_{t \to \alpha} (t-\alpha)^{\frac{1}{2}} \tau_{1yz}^{-}(t)$$

the condition that for the elastic–plastic case $k = 0$ gives

$$\frac{a}{\alpha} = \cos\left(\frac{\pi}{2}\frac{q_0}{\tau_0}\right). \tag{29}$$

Here we note that because of symmetry the result is independent of the material properties and is the same as that found earlier by using the dislocation approach [16]. To compare this simple result with a known solution based on the field approach to plasticity obtained in [12], where a numerical solution for an edge crack is given, we observe that in the example under consideration, again because of symmetry, the yz-plane is stress-free. Hence (29) also provides a solution for a semi-infinite medium with an edge crack of length a and subjected to constant shear, $\tau_{yz}^{\infty} = q_0$, at infinity (or the torsion of a circular bar containing a longitudinal surface crack the depth of which is very small compared to the diameter of the bar). The results reproduced from Fig. 2 of [12] and (29) are shown in Fig. 3. The apparent agreement found for this example indicates that the

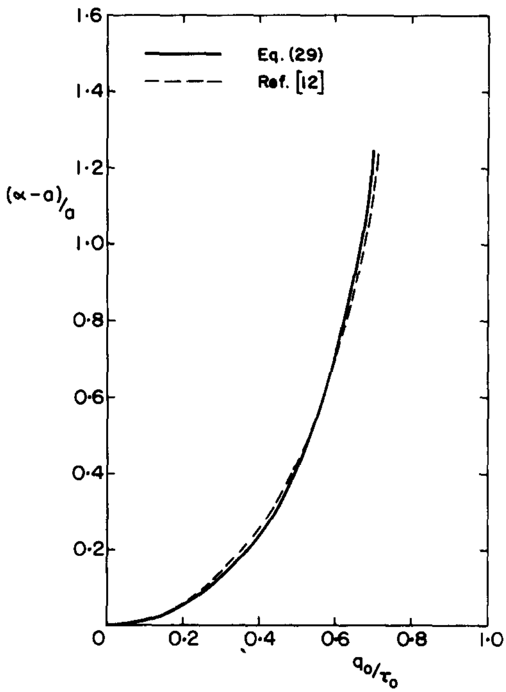

FIG. 3. Size of the yield zone in the plane of the crack.

simple technique described above, which is essentially based on dislocations, may be used to estimate the plastic zone size.*

* For a similar approach to calculate the plastic zone size in plane problems and comparison with experimental results, see [23].

If we define the plastic zone size as $p = \alpha - a$, for $q_0 \ll \tau_0$ (or in the more general case of relatively small external loads leading to $p \ll a$) from (29) we may write

$$p \cong \left(\frac{\pi}{2\tau_0}\right)^2 k^2. \tag{30}$$

Since in most problems the elastic–plastic analysis is too complicated to lead to a closed form solution for p, equation (30), which requires only the elastic solution, may be of considerable practical importance in the estimation of the plastic zone size.

In the application of the results to fracture of materials a quantity which plays an important role is the displacement at the tip of the crack. It has been shown that a critical value of this displacement multiplied by the yield stress in shear is equivalent to the (plastic) work required to increase the area of the crack by a unit length in the Griffith–Irwin–Orowan theory of quasi-brittle fracture [16, 17, see also the discussion at the end of this paper]. In the bonded media the relative displacement along the crack and in the plastic zone may be obtained from

$$w_1^- - w_2^+ = h(t) = \int \frac{\mu_1 + \mu_2}{2\mu_2} [G^-(t) - G^+(t)]\, dt \tag{31}$$

with the appropriate integration constant insuring that $h(\pm\alpha) = 0$. In (31) $G(\zeta)$ is the total solution (for S^-), i.e. it is the sum of the solutions due to the external loads and τ_0 acting along the yield zones. For the example under consideration we obtain

$$h(t) = -\frac{\tau_0}{\pi}\frac{\mu_1+\mu_2}{\mu_1\mu_2} 2\alpha \left[\left(\frac{1}{c^2+1} - \frac{1}{\theta^2+1}\right)\log\frac{\theta - c}{\theta + c} + \left(\frac{c^2}{c^2+1} - \frac{\theta^2}{\theta^2+1}\right)\right.$$

$$\left. .\log\frac{c\theta+1}{c\theta-1}\right] \quad c = [(\alpha+a)/(\alpha-a)]^{\frac{1}{2}}, \quad \theta = [(\alpha+t)/(\alpha-t)]^{\frac{1}{2}}. \tag{32}$$

For $(q_0/\tau_0) = \frac{2}{3}$ the variation of the relative displacement $h(t)$ with t/a and μ_2/μ_1 is shown in Fig. 4. In the more general case of non-symmetrical loading the calculation of α as well as $h(t)$ may be somewhat laborious. If q_0/τ_0 is sufficiently small, in this case too the concept of strength of the stress singularity may be effectively used. For this we first

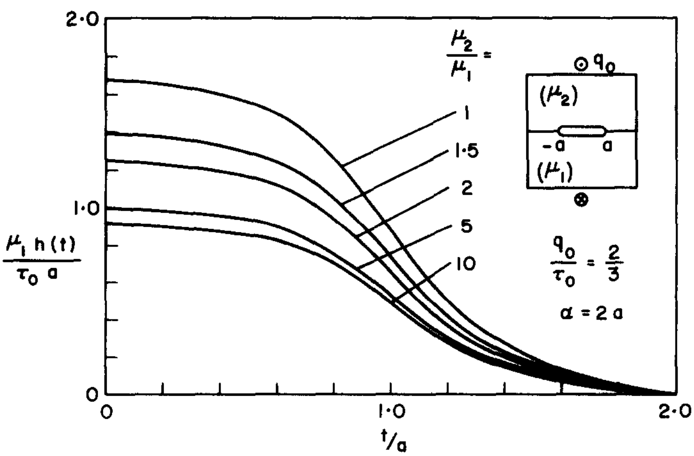

FIG. 4. Relative displacement on the crack surface and the yield zone.

note that the important quantity employed in the fracture criterion is the value of the relative displacement at the crack tip which for the present example is found to be*

$$h(a) = -2a\frac{\tau_0}{\pi}\frac{\mu_1+\mu_2}{\mu_1\mu_2}\log\frac{\alpha}{a}. \tag{33}$$

For small values of q_0/τ_0 from (33), (29) and (30) an approximation to the absolute value of crack tip displacement may be obtained as

$$h(a) \simeq \frac{2}{\pi}p\tau_0\frac{\mu_1+\mu_2}{\mu_1\mu_2} = \frac{\pi}{2\tau_0}\frac{\mu_1+\mu_2}{\mu_1\mu_2}k^2 \tag{34}$$

where the stress intensity factor, k, at $t = a$ is defined by

$$k = \lim_{t\to a}\sqrt{(t-a)}\,\tau_{1yz}^-(t).$$

Figure 5 shows the variation of $h(a)$ in μ_2/μ_1 and q_0/τ_0. The figure also shows the approximate value of $h(a)$ obtained from (34) and the relative error. Thus, for example

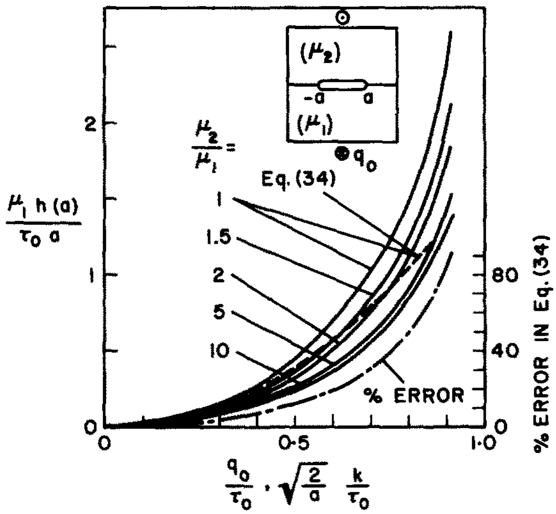

FIG. 5. Crack tip displacement.

if a relative error of 10 per cent is permissible in the evaluation of $h(a)$, up to $q_0/\tau_0 = 0{\cdot}46$ (34) may be used to calculate $h(a)$ without going through a lengthy process of obtaining the plastic zone size. It may easily be shown that for $\mu_1 = \mu_2$ the foregoing results reduce to those obtained in [16].

As a second example we consider the torsion of a cylindrical bar containing a semi-circular longitudinal groove with a radial crack at its bottom. Assuming that the transverse dimensions of the bar is large compared to the radius of the groove and the depth of the crack, the problem may be considered one of longitudinal shear of a semi-infinite medium or an infinite medium with a circular hole and two symmetric radial cracks (Fig. 6). If only the symmetric loading is considered the stresses for homogeneous medium

* The minus sign in (32) and (33) is due to the sign convention of displacement and has no significance in applications.

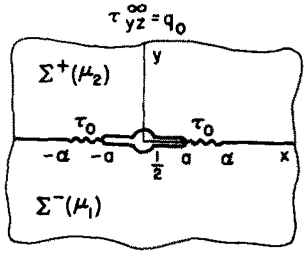

FIG. 6. Bonded plane with circular cavity and radial cracks.

and dissimilar media will be the same. Without a loss in generality, taking the diameter of the circle to be unity, the mapping function and its derivative are given by

$$\eta = \omega(\zeta) = \tfrac{1}{2}[\zeta+\sqrt{(\zeta^2-1)}]$$

$$|\omega'(\zeta)| = \begin{cases} \dfrac{1}{2\sqrt{(1-t^2)}}, & |t| < 1 \\ \dfrac{1}{2}\left[1+\dfrac{t}{\sqrt{(t^2-1)}}\right], & |t| > 1. \end{cases} \tag{35}$$

If $\pm a$ and $\pm\alpha$ correspond to the crack tips and ends of the plastic zone in the real medium (Fig. 6), the corresponding quantities in the equivalent shear problem, (i.e. in the ζ plane) will be

$$b = \frac{4a^2+1}{4a}, \qquad \beta = \frac{4\alpha^2+1}{4\alpha}, \qquad t = \frac{4r^2+1}{4r}. \tag{36}$$

First consider the elastic problem with cracks extending to $\pm\alpha$ and external loads consisting of the shear stress at infinity $\tau_{1s\sigma}^{\infty} = \tau_{2s\sigma}^{\infty} = q_0$. The solution of equivalent shear problem is given by the previous example (where β replaces a in (27)). From (21), (27), (35) and (36) the contact stress, and the stress intensity factor are obtained as

$$\tau_{1\sigma s}^{-}(r) = \frac{\alpha q_0(16r^4-1)}{4r^2[(16\alpha^2r^2-1)(r^2-\alpha^2)]^{\frac{1}{2}}}, \qquad |r| > |\alpha|$$
$$k_1 = \frac{q_0}{4\alpha}\left(\frac{16\alpha^4-1}{2\alpha}\right)^{\frac{1}{2}}. \tag{37}$$

Note that for the case of a hole without any cracks, $\alpha = \frac{1}{2}$ and the contact stress and stress concentration factor become

$$\tau_{1s\sigma}^{-}(r) = \frac{q_0(4r^2+1)}{4r^2}, \qquad \frac{\tau_{1s\sigma}^{-}(\frac{1}{2})}{q_0} = 2.$$

Considering now the shear problem with the external loads $\tau_{1s\sigma}^{-} = \tau_{2s\sigma}^{+} = \tau_0$ (τ_0 being the yield stress in shear) acting on the part of the crack surface $a < |r| < \alpha$, in the equivalent shear problem the corresponding surface tractions will be (see: (21))

$$g_1(t) = g_2(t) = \tau_0|\omega'(t)| \qquad b < |t| < \beta$$
$$= 0 \qquad |t| < b.$$

Observing that for this loading condition $p(t)$ and $M(\zeta)$ are zero, from (15) we obtain

$$G(\zeta) = -\frac{\tau_0}{2\pi\mu_1 i}\left\{\left[\frac{2\zeta}{\sqrt{(\zeta^2-\beta^2)}}\left(\frac{\pi}{2}-\arcsin\frac{b}{\beta}+\arctan\left(\frac{\beta^2-b^2}{b^2-1}\right)^{\frac{1}{2}}\right)\right.\right.$$
$$+\arctan\left(\frac{(\beta+b)(\zeta-\beta)}{(\beta-b)(\zeta+\beta)}\right)^{\frac{1}{2}}-\arctan\left(\frac{(\beta+b)(\zeta+\beta)}{(\beta-b)(\zeta-\beta)}\right)^{\frac{1}{2}}$$
$$\left.\left.-\frac{2\zeta}{\sqrt{(\zeta^2-1)}}\arctan\left(\frac{(\beta^2-b^2)(\zeta^2-1)}{(b^2-1)(\zeta^2-\beta^2)}\right)^{\frac{1}{2}}\right]\right\}$$

from which the contact stress and the stress intensity factor are found to be

$$\tau_{1\sigma s}^{-}(r) = \frac{\tau_{1yz}^{-}(t)}{|\omega'(t)|} = -\frac{\tau_0\sqrt{(t^2-1)}}{\pi[t+\sqrt{(t^2-1)}]}\left\{\frac{2t}{\sqrt{(t^2-\beta^2)}}\left[\frac{\pi}{2}-\arcsin\frac{b}{\beta}\right.\right.$$
$$\left.+\arctan\left(\frac{\beta^2-b^2}{b^2-1}\right)^{\frac{1}{2}}\right]+\arctan\left[\frac{(\beta+b)(t-\beta)}{(\beta-b)(t+\beta)}\right]^{\frac{1}{2}}$$
$$\left.-\arctan\left[\frac{(\beta+b)(t+\beta)}{(\beta-b)(t-\beta)}\right]^{\frac{1}{2}}-\frac{2t}{\sqrt{(t^2-1)}}\arctan\left[\frac{(\beta^2-b^2)(t^2-1)}{(b^2-1)(t^2-\beta^2)}\right]^{\frac{1}{2}}\right\}$$

$$k_2 = -\frac{\tau_0\sqrt{(16\alpha^4-1)}}{4\pi\alpha\sqrt{(2\alpha)}}\left\{\frac{\pi}{2}-\arcsin\frac{\alpha(1+4a^2)}{a(1+4\alpha^2)}+\arctan\frac{[(\alpha^2-a^2)(16\alpha^2a^2-1)]^{\frac{1}{2}}}{\alpha(4a^2-1)}\right\}.$$

The condition that for the elastic–plastic deformation $k = k_1+k_2 = 0$ gives the following equation which determines the unknown constant α

$$\frac{q_0}{\tau_0} = \frac{1}{\pi}\left\{\frac{\pi}{2}-\arcsin\frac{\alpha(4a^2+1)}{a(4\alpha^2+1)}+\arctan\frac{[(\alpha^2-a^2)(16\alpha^2a^2-1)]^{\frac{1}{2}}}{\alpha(4a^2-1)}\right\}. \quad (38)$$

If there are no radial cracks emanating from the circle, $a = \frac{1}{2}$ and (38) can be solved for α explicitly, giving

$$\alpha = \tfrac{1}{2}\tan\frac{\pi q_0}{2\tau_0}$$

where $\alpha-\frac{1}{2}$ is now the depth of the plastic zone in the slip plane $y = 0$ around a circular hole of radius $\frac{1}{2}$.

Further results and examples where the elastic and elastic–plastic solutions are dependent on the shear moduli of the adjoining media are given in the Appendix.

5. DISCUSSION AND CONCLUSIONS

The results of this paper are intended for use in the fracture of bonded dissimilar materials with cracks or cavities under anti-plane shear. The fracture criterion proposed in [16] and [17] namely that the fracture will begin when the "displacement" at the root of the notch reaches to a critical value, may also be used for bonded materials. For small values of q_0/τ_0 or p/a, p and a being the plastic zone size and the half crack length, this criterion is equivalent to the Irwin–Orowan modification of Griffith theory. In the Griffith theory for anti-plane shear the criterion is

$$q_0 = \left(\frac{4\mu\gamma}{\pi a}\right)^{\frac{1}{2}} \quad \text{or} \quad k = \left(\frac{2\mu\gamma}{\pi}\right)^{\frac{1}{2}} = \text{constant} \tag{39}$$

where γ is the specific surface energy and the criterion may be stated as "when the stress intensity factor reaches to a critical value, which is a material parameter, the fracture will begin". In the Irwin–Orowan modification of the theory 2γ stands for the (plastic) work required to increase the area of the crack by a unit amount. On the other hand (34) may be rewritten as

$$q_0 = \left(\frac{4}{\pi a}\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\frac{\tau_0 h(a)}{2}\right)^{\frac{1}{2}} \quad \text{or} \quad k = \left(\frac{2}{\pi}\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\frac{\tau_0 h(a)}{2}\right)^{\frac{1}{2}}. \tag{40}$$

Comparison of (39) and (40) indicates that if one assumes a critical value of $h(a)$ to be a material parameter the two theories are identical, $\tau_0 h(a)$ corresponding to the plastic work and $(2\mu_1\mu_1)/(\mu_1+\mu_2)$ replacing μ in the Irwin–Orowan extension of Griffith theory. It should be noted that the second equation in (40) is the more general statement of the criterion and the stress intensity factor k in addition to being linearly dependent on the external loads may also depend on μ_1 and μ_2. The critical notch-root-displacement criterion also demonstrates the importance of the magnitudes of μ_1 and μ_2 in the fracture along the interface. From (33) it is seen that if μ_1, μ_2 are very high a large plastic zone will be required for $h(a)$ to reach a critical value, which means that q_0 will approach τ_0. Of course in the limiting case of bonded rigid media there will be no notch effect.

From (19) it is seen that in bonded dissimilar media with cracks the singularities are ordinary branch points and the stresses are of the form

$$\tau_{ij} = \rho^{-\frac{1}{2}} A_{ij}(\theta, \mu_1, \mu_2)$$

where ρ, θ are the polar coordinates in xy-plane with origin at the tip and A_{ij} are bounded functions. On the other hand it was shown that [6] the singularities caused by the xy-components of the external loads (i.e. plane strain or generalized plane stress) are essential singularities and stresses are of the form

$$\sigma_{ij} = \rho^{-\frac{1}{2}} B_{ij}(\theta)\sin\left(\gamma\log\frac{\rho}{a}\right) + \rho^{-\frac{1}{2}} C_{ij}(\theta)\cos\left(\gamma\log\frac{\rho}{a}\right)$$

where γ is a bi-elastic constant and a is a finite length. It is seen that even though the strength of singularities is the same for both loading conditions, the oscillation phenomenon encountered in plane extensional problems is not observed in longitudinal shear.

In anti-plane problems for dissimilar media, if the external loads are symmetric with respect to xz-plane, the contact stress is independent of the shear moduli meaning that, unlike the results found for plane problems [6, 7], the stresses and displacements in one medium are unaffected by the elastic properties of the other. However, this is not the case if the symmetry does not exist (see the examples (b), (d) and (e) in the Appendix).

The solutions given by (9) and (15) and their variations for the case of cavities apply without modification to the bonded quarter plane problems, $(x > 0)$, and hence to the torsion of cylindrical bars transverse dimensions of which are large compared to the dimensions of the imperfections. The solution of this problem is the same as that of bonded semi-infinite media where both the external loads and the geometry are symmetric with respect to the yz-plane. If there are external loads acting on the surfaces of the quarter planes, $x = 0$, the problem may be solved by using the results similar to those of the example (d) in the Appendix as the Green's function (where T is twice the magnitude of the load acting on the quarter plane).

The elastic results found in this paper are applicable without modification to the problem of heat conduction in dissimilar materials with the same geometry where w, τ and μ directly correspond to the temperature distribution, the heat flux and the coefficient of heat conduction, respectively.

Similarity of the results* of elastic–plastic analysis in anti-plane and plane problems leads one to cautiously conjecture that in the absence of solutions for plane problems (which are considerably more difficult), the solutions for anti-plane problems with the same geometry may be used for rough estimates of stress intensity factors and plastic zone sizes in the symmetric mode of plane problems.

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APPENDIX

Further Examples

(a) Single bond, $L = (-a, a)$, T at ∞ (Fig. 7)

$$\Omega_1(\zeta) = -\frac{T}{\pi\mu_1\sqrt{(\zeta^2-a^2)}}, \qquad \tau_{1yz}^{-}(t) = \frac{T}{\pi\sqrt{(a^2-t^2)}}, \qquad |t| < a.$$

(b) $L = (-a, a)$, $\tau_{1xz}^{\infty} = p_1$, $\tau_{2xz}^{\infty} = p_2$

$$\Omega_1(\zeta) = \frac{p_1}{\mu_1} + \frac{\mu_2}{\mu_1+\mu_2}\left(\frac{p_2}{\mu_2}-\frac{p_1}{\mu_1}\right)\left[1-\frac{\zeta}{\sqrt{(\zeta^2-a^2)}}\right]$$

* In fact the expression for the plastic zone size given by (29) is identical to that obtained in [23] for the plane extensional problems which is the only available solution using a similar procedure. See also the footnote in the Appendix.

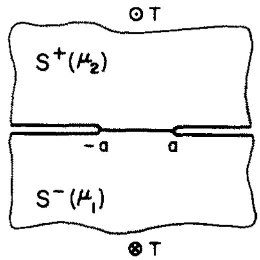

FIG. 7 Bonded plane with edge notches.

$$\tau_{1yz}^{-}(t) = \frac{\mu_1 p_2 - \mu_2 p_1}{\mu_1 + \mu_2} \frac{t}{\sqrt{(a^2 - t^2)}}, \qquad |t| < a.$$

(c) $L = (-a, a), \qquad g_1(t) = g_2(t) = T\delta(t - t_0), \qquad t_0 > a$

$$\Omega_1(\zeta) = \frac{T\sqrt{(t_0^2 - a^2)}}{\pi\mu_1}\left\{\frac{1}{(t_0 - \zeta)\sqrt{(\zeta^2 - a^2)}} - \frac{2}{a^2}\left[1 - \frac{\zeta}{\sqrt{(\zeta^2 - a^2)}}\right]\right\}$$

$$\tau_{1yz}^{-}(t) = \frac{T\sqrt{(t_0^2 - a^2)}}{\pi\sqrt{(a^2 - t^2)}}\left(\frac{2t}{a^2} + \frac{1}{t_0 - t}\right), \qquad |t| < a.$$

(d) $L = (-a, a)$, T at $\zeta_0 \in S^-, \qquad w_2(\infty) = 0$ (Fig. 8)

$$\Omega_1(\zeta) = \frac{T}{\pi(\mu_1 + \mu_2)}\left\{\frac{1}{\sqrt{(\zeta^2 - a^2)}} - \frac{1}{2}\left(\frac{1}{\zeta - \zeta_0} + \frac{1}{\zeta - \bar{\zeta}_0}\right)\right.$$
$$\left. + \frac{\mu_2}{2\mu_1\sqrt{(\zeta^2 - a^2)}}\left[\frac{\sqrt{(\zeta_0^2 - a^2)}}{\zeta_0 - \zeta} + \frac{\sqrt{(\bar{\zeta}_0^2 - a^2)}}{\bar{\zeta}_0 - \zeta}\right]\right\}$$

$$\tau_{1yz}^{-}(t) = \frac{-\mu_1 T}{2\pi(\mu_1 + \mu_2)\sqrt{(a^2 - t^2)}}\left\{2 - \frac{\mu_2}{\mu_1}\left[\frac{\sqrt{(\zeta_0^2 - a^2)}}{t - \zeta_0} + \frac{\sqrt{(\bar{\zeta}_0^2 - a^2)}}{t - \bar{\zeta}_0}\right]\right\}.$$

For $\zeta_0 = ib$ and yield stress in shear $= \tau_0$. α is found from

$$\frac{T\mu_2}{(\mu_1 + \mu_2)\tau_0} = \frac{\sqrt{(\alpha^2 + b^2)}\log\{[\sqrt{(a^2 - \alpha^2)} + a]/\alpha\}}{b + (\mu_1/\mu_2)\sqrt{(\alpha^2 + b^2)}}$$

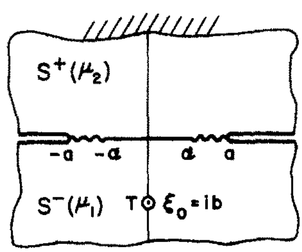

FIG. 8. Bonded plane with edge notches under concentrated load.

(e) $L = (-a, a)$, $g_1(t) = T\delta(t - t_0)$, $g_2(t) = 0$, $w_2(\infty) = 0$, $(t_0 > a)$

$$\Omega_1(\zeta) = \frac{T}{\pi(\mu_1+\mu_2)}\left[-\frac{1}{\zeta-t_0}+\frac{1}{\sqrt{(\zeta^2-a^2)}}-\frac{\mu_2\sqrt{(t_0^2-a^2)}}{\mu_1(\zeta-t_0)\sqrt{(\zeta^2-a^2)}}\right]$$

$$\tau_{1yz}^-(t) = -\frac{\mu_1 T}{\pi(\mu_1+\mu_2)\sqrt{(a^2-t^2)}}\left[1+\frac{\mu_2}{\mu_1}\frac{\sqrt{(t_0^2-a^2)}}{t_0-t}\right].$$

(f) Two bonds, $L = (-b, -a) + (a, b)$, T at ∞

$$\Omega_1(\zeta) = \frac{-T\zeta}{\pi\mu_1[(\zeta^2-a^2)(\zeta^2-b^2)]^{\frac{1}{2}}}$$

$$\tau_{1yz}^-(t) = \frac{Tt}{\pi[(t^2-a^2)(b^2-t^2)]^{\frac{1}{2}}}, \qquad (t \in L).$$

(g) Two cracks, $L' = (-b, -a)+(a, b)$, $\tau_{1yz}^\infty = \tau_{2yz}^\infty = q_0$

$$G(\zeta) = -\frac{iq_0}{\mu_1}\left[\zeta^2-b^2\frac{E(m)}{K(m)}\right]\frac{1}{\sqrt{[(\zeta^2-a^2)(\zeta^2-b^2)]}}, \qquad m^2 = \frac{b^2-a^2}{b^2}$$

$$\tau_{1yz}^-(t) = \mp q_0\left[t^2-\frac{b^2E(m)}{K(m)}\right]\frac{1}{\sqrt{[(t^2-a^2)(t^2-b^2)]}} \qquad \begin{pmatrix}+, |t|>b \\ -, |t|<a\end{pmatrix}$$

Stress intensity factors:*

$$k(a) = q_0\left[\frac{b^2E(m)}{K(m)}-a^2\right]\frac{1}{\sqrt{[2a(b^2-a^2)]}}$$

$$k(b) = q_0b^2\left[1-\frac{E(m)}{K(m)}\right]\frac{1}{\sqrt{[2b(b^2-a^2)]}}.$$

(h) Circular cavity of radius $\frac{1}{2}$, concentrated shear loads $\mp Q$ at $\frac{1}{2}e^{\pm i\theta_0}$

$$\tau_{1s\sigma}^-(r) = -\frac{2Q}{\pi(1+4r^2-4r\cos\theta_0)}, \qquad (|r|>\tfrac{1}{2}).$$

(i) Elliptic cavity, $\tau_{1yz}^\infty = \tau_{2yz}^\infty = q_0$

$$\tau_{1\sigma s}(r(t)) = \frac{2q_0t[t+\sqrt{(t^2-1)}]}{1-m+2\sqrt{(t^2-1)}[t+\sqrt{(t^2-1)}]}, \qquad (t > 1)$$

$m = (a-b)/(a+b)$, a, b semi-axes, a is normalized to be $(1+m)/2$,

$$r(t) = t-\frac{1-m}{2[t+\sqrt{(t^2-1)}]}.$$

Stress concentration factor,

$$\frac{\tau_{1\sigma s}^-(a)}{q_0} = 1+\frac{a}{b}.$$

(j) Elliptic cavity with cracks along $a < |r| < c$, $\tau_{1yz}^\infty = \tau_{2yz}^\infty = q_0$ (Fig. 9)

* Note that $k(a)$ is always larger than $k(b)$ and observe the similarity (in fact, for the symmetric mode, the identity) of the results to those for the plane problem [24].

$$\tau_{1\sigma s}^{-}(r) = \frac{2q_0 t[t+\sqrt{(t^2-1)}]\sqrt{(t^2-1)}}{\{1-m+2[(t+\sqrt{(t^2-1)})]\sqrt{t^2-1}\}\sqrt{(t^2-\gamma^2)}}, \qquad t > \gamma$$

$$r(t) = t - \frac{1-m}{2[t+\sqrt{(t^2-1)}]}, \qquad c = \gamma - \frac{1-m}{2[\gamma+\sqrt{(\gamma^2-1)}]}.$$

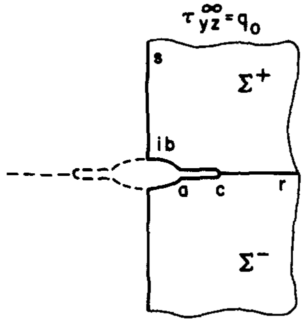

FIG. 9. Bonded plane with elliptic cavity and cracks.

(k) Torsion of a cylindrical bar with a V-shaped groove of angle $\pi/2$ where a is small compared to transverse dimensions (Fig. 10)

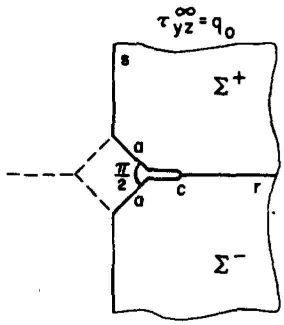

FIG. 10. Bonded half-plane with V-shaped groove and crack.

$$\tau_{1\sigma s}^{-}(r) = \frac{q_0 t^{\frac{1}{2}}}{(t^2-b^2)^{\frac{1}{4}}}, \qquad (t > b)$$

$$b = 2a\sqrt{\left(\frac{2}{\pi}\right)}\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}, \qquad r = \frac{a}{\sqrt{2}} + \int_b^t \frac{t^{\frac{1}{2}}\,dt}{(t^2-b^2)^{\frac{1}{4}}}.$$

In the neighborhood of the apex, $r = a/\sqrt{2}$:

$$\tau_{1\sigma s}^{-}(r) \simeq q_0\left[\frac{2b}{3\left(r-\frac{a}{\sqrt{2}}\right)}\right]^{\frac{1}{3}}.$$

If the bar contains a crack along $a/\sqrt{2} < r < c$:

$$\tau_{1\alpha s}^{-}(r) = \frac{q_0\sqrt{t}\,(t^2-b^2)^{\frac{1}{2}}}{\sqrt{(t^2-\gamma^2)}}, \qquad t > \gamma$$

$$c = \frac{a}{\sqrt{2}} + \int_b^\gamma \frac{\sqrt{t}\,\mathrm{d}t}{(t^2-b^2)^{\frac{1}{2}}}.$$

Stress intensity factor:

$$k = \lim_{r\to c}\sqrt{(r-c)}\tau_{1\alpha s}^{-}(r) = \frac{q_0}{\sqrt{2}}\gamma^{\frac{1}{2}}(\gamma^2-b^2)^{\frac{1}{4}}.$$

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Résumé—Le problème de deux milieux semi-infinis attachés, ayant des propriétés différentes, contenant des fissures ou cavités sur la tranche et sujets à des forces de cisaillement longétudinaux de divers côtés, a été considéré. La solution élastique dans laquelle les composants-z du forces externe sont uniquement pris en considération, complète le problème de résistance plane où les forces externes se trouvent dans un plan-xy. Employant l'approche de la théorie des dislocations aux déformations plastiques, une simple technique basée sur la théorie élastique est développée, afin d'estimer la taille de la zone plastique au voisinnage de la fissure et des cavités. Quelques exemples sont expliqués et une liste de solutions donnée dans l'annexe.

Zusammenfassung—Das Problem von zwei verbundenen einseitig-unbegrenzten Medien mit verschiedenen Eigenschaften, welche Risse oder Aushöhlungen an der Grenzfläche enthalten, und welche zu Längsschubbelastungen in verschiedenen Wegen ausgesetzt sind, wird untersucht. Die elastische Lösung in welcher nur die z-Komponenteder äusseren Belastung in Betracht gezogen wird, ergänzt das Flächenbeaspruchungs Problem, in welchem die äusseren Belastungen in der xy-Fläche liegen. Unter Verwendung der Annäherung der Versetzungs theorie zu plastichen Verformungen, ein einfaches Verfahren, welches auf der elastischen Theorie beruht wird entwickelt, um die Grösse der plastischen Zone in der Nähe von Rissen und Aushöhlungen zu schätzen. Einige Beispiele sind ausgearbeitet und eine Liste von Lösungen ist im Anhang gegeben.

Абстракт—Рассматриваются различные возможности проблемы двух связанных семи-бесконечных материалов с различными свойствами, содержащих трещины или полости на поверхности раздела и подвергнутых нагрузкам продольного сдвига. Упругое Ретение, в котором принят во внимание только зед-компонент внешних нагрузок, дополняет проблему деформации плоскости, где внещние нагрузки лежат в икс-плоскости. Применяя подход теории дислокации для пластических деформаций, разработана простая техника, основанная на теории упругости для учёта величины пластичной зоны вблизи трещин и полостей. Выработаны некоторые примеры и в приложении дан список решений.